Clustering in Random Pattern-Avoiding Permutations

A *cluster* of length l in a permutation from S_n is a set of l consecutive numbers that appear in any order in l consecutive positions in the permutation. For $n \geq l \geq 2$ and $\tau \in S_l$, let $N_l^{(n)}(\sigma)$ denote the number of clusters of length l in σ, and let $N_{l;\tau}^{(n)}(\sigma)$ denote the number of clusters of length l whose order is the pattern τ. (For example, $\sigma = 317564928 \in S_9$ has the cluster 7564 of length $l = 4$ and pattern $\tau = 4231$.)

For $\eta \in S_3$, let $S_n^{\text{av}(\eta)}$ denote the set of permutations in S_n that avoid the pattern η, and let $E_n^{\text{av}(\eta)}$ denote the expectation with respect to the uniform probability measure on $S_n^{\text{av}(\eta)}$. We obtain explicit formulas for $E_n^{\text{av}(\eta)} N_{l;\tau}^{(n)}$ and $E_n^{\text{av}(\eta)} N_l^{(n)}$. These exact formulas yield asymptotic formulas as $n \to \infty$ with l fixed, and also with $l = l_n \to \infty$.

In particular, for fixed l, depending on η and τ, the expectation $E_n^{\text{av}(\eta)} N_{l;\tau}^{(n)}$ either grows linearly in n or is bounded and bounded away from zero. The expectation $E_n^{\text{av}(\eta)} N_l^{(n)}$ always has linear growth in n, when l is fixed. For certain $\eta \in S_3$, $\lim_{n\to\infty} E_n^{\text{av}(\eta)} N_{l_n}^{(n)}$ is equal to infinity or zero depending on whether l_n is on an order less than $n^{\frac{2}{3}}$ or greater than $n^{\frac{2}{3}}$, while for other $\eta \in S_3$, this dichotomy occurs depending on whether $\lim_{n\to\infty}(l_n - \frac{\log n}{\log 4})$ is equal to $-\infty$ or $+\infty$.

Analogous results are obtained for $S_n^{\text{av}(\eta_1,\cdots,\eta_r)}$, the permutations in S_n simultaneously avoiding the patterns $\{\eta_i\}_{i=1}^r$, in the case that $\{\eta_i\}_{i=1}^n$ are all *simple* permutations. A particular case of this, where we can calculate the asymptotic behavior explicitly, is the set of *separable* permutations, which corresponds to $r = 2$, $\eta_1 = 2413, \eta_2 = 3142$.